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A Computational Procedure for High-Order Adiabatic Invariants

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When the frequency of the harmonic oscillator is slowly varying in time, the invariants can be expanded in powers of a small parameter characterizing the slowness of the variation. It is the purpose of this paper to present a recurrence procedure yielding, with the help of an algebraic computer program, the terms of the adiabatic invariants determined up to the tenth order. The obtained formulas are checked over with two examples.

1. INTRODUCTION

When the parameters of a physical system are slowly varying under the effect of external perturbations, some quantities are constant at any order of a small parameter ϵ characterizing the slowness of the variation. Of course, this does not imply that these quantities are exactly constant but that their variation goes to zero faster than any power of ϵ. Such quantities are called adiabatic invariants.

To find the series of adiabatic invariants for the harmonic oscillator a procedure has been proposed by Kulsrud [1] and later by Lewis [2], who determined a class of exact invariants. In this paper we suggest an iterative procedure derived from Chandrasekhar's method [3], which has been shown to be equivalent to the Lewis method [4]. A recurrence formula is obtained and used to get the different terms of the expansion with an algebraic computer language (FORMAC). The numerical value of the adiabatic invariants can be obtained up to order ten.

Two examples are given and discussed. In the first case the adiabatic series converges strictly to the exact invariant. In the second case a nonadiabatic jump cannot be avoided and the series, which looks numerically like an asymptotic one, gives a slightly modified value of the invariant. Values of ϵ are found which are small enough for the nonadiabatic jump to be quite negligible but big enough to exhibit an improvement when high order invariants are considered.

2. Analytic Treatments of Adiabatic Invariants

Consider the general equation of the linear oscillator,

$$\frac{d^2q}{dt^2} + \omega^2(t)\,q = 0, \tag{1}$$

where the Hamiltonian for unit mass is

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) \tag{2}$$

and q and p are the conjugate coordinate and momentum.

In the case of a slowly varying frequency, the ratio, energy/frequency, is well known to be the zero order adiabatic invariant, the first and second order adiabatic invariant are

$$L_1 = K_0 + \epsilon K_1 \qquad \text{and} \qquad L_2 = K_0 + \epsilon K_1 + \epsilon^2 K_2\,,$$

respectively, where:

$$\begin{aligned} K_0 &= E/\omega, \\ \epsilon K_1 &= \frac{d\omega}{dt}\frac{qp}{2\omega^2}\,, \\ \epsilon^2 K_2 &= \frac{q^2}{8\omega^3}\left(\frac{d\omega}{dt}\right)^2 + \frac{1}{8\omega^4}\left[\frac{d^2\omega}{dt^2} - \frac{3}{2\omega}\left(\frac{d\omega}{dt}\right)^2\right], \end{aligned} \tag{3}$$

as first derived by Kulsrud [1].

In Eq. (3) the parameter ϵ appears automatically if we take $\omega = \omega(u)$ with $u = \epsilon t$. But the method given by Kulsrud is of no use to compute high order invariants because of the difficulty of the calculus. Two other methods can be used to find the adiabatic series.

A. *Lewis' Method*

To solve Eq. (1) Lewis introduced a new variable and a new function:

$$q = Qw, \tag{4}$$

$$\theta = \int_0^t \frac{1}{w^2(t')}\, dt', \tag{5}$$

Eq. (1) is, therefore, equivalent to the system,

$$\begin{cases} \dfrac{d^2Q}{d\theta^2} + Q = 0, \\ \dfrac{d^2w}{dt^2} + \omega^2(t)\, w = w^{-3}, \end{cases} \tag{6}$$

consequently,

$$I = \frac{1}{2}\left[Q^2 + \left(\frac{dQ}{d\theta}\right)^2\right] \tag{7}$$

is an *exact* invariant for the motion.

Taking into account Eq. (4), Eq. (7) becomes

$$I = \frac{1}{2}\left[\frac{q^2}{w^2} + \left(q\,\frac{dw}{dt} - w\,\frac{dq}{dt}\right)^2\right]. \tag{8}$$

As any initial condition on w, (dw/dt) may be taken as Eq. (8) defines a family of exact invariants.

Now, assuming that ω is a slowly varying function of time and taking $\omega = \omega(u = \epsilon t)$ the differential equation for w becomes

$$\epsilon^2 \frac{d^2w}{du^2} + \omega^2(u)\, w = w^{-3}, \tag{9}$$

so that a perturbation method can be used to solve Eq. (9). We look for a solution to the form,

$$w = w_0 + \epsilon^2 w_2 + \cdots + \epsilon^{2n} w_{2n} + \cdots. \tag{10}$$

The zero order solution of w, namely w_0, is easily found,

$$w_0 = \omega^{-1/2}. \tag{11}$$

Introducing Eq. (11) in Eq. (8) we see that the zero order adiabatic invariant is

$$I_0 = \frac{1}{2}\left[q^2\omega + \frac{1}{\omega}\left(\frac{dq}{dt}\right)^2\right] + \epsilon\,\frac{q}{2\omega^2}\cdot\frac{dq}{dt}\cdot\frac{d\omega}{du} + \epsilon^2\,\frac{q^2}{8\omega^3}\left(\frac{d\omega}{du}\right)^2. \tag{12}$$

Equation (12) shows that the two first terms and a part of the third term of the Kulsrud series are recovered. Unfortunately this last method is difficult to use for solving the problem of high order invariants.

B. *Chandrasekhar's Method*

Following Chadrasekhar [3] let us introduce in Eq. [1] a new variable t_1 and a new function q_1 given by

$$\begin{cases} \dfrac{dt_1}{dt} = \omega, \\ q_1 = \omega^{1/2} q, \end{cases} \tag{13}$$

which gives

$$\frac{d^2 q_1}{dt_1{}^2} + \omega_1{}^2 q_1 = 0, \tag{14}$$

with

$$\omega_1{}^2 = 1 + \omega^{-3/2} \frac{d^2}{dt^2} (\omega^{-1/2}). \tag{15}$$

Again, it is useful to introduce explicitly the slowness of the time variation of ω through the change of variable $u = \epsilon t$. Equation (15) is changed in

$$\omega_1{}^2 = 1 + \epsilon^2 \omega^{-3/2} \frac{d^2}{du^2} (\omega^{-1/2}). \tag{16}$$

This procedure may be repeated and the general expressions,

$$\begin{cases} \dfrac{dt_{k+1}}{dt_k} = \omega_k , \\ q_{k+1} = \omega_k^{1/2} q_k , \end{cases} \tag{17}$$

yield

$$\frac{d^2 q_{k+1}}{dt_{k+1}^2} + \omega_{k+1}^2 q_{k+1} = 0, \tag{18}$$

with

$$\omega_{k+1}^2 = 1 + \omega_k^{-3/2} \frac{d^2}{dt_k{}^2} (\omega_k^{-1/2}).$$

As an ϵ^2 factor appears at each step in the second term of Eq. (16), for $k = n$, ω_{k+1}^2 can be taken strictly equal to 1, which closes the iterative process.

Now the product,

$$\rho_n = (\omega\omega_1 \dots \omega_n)^{-1/2}, \tag{19}$$

has been shown [4] to be equivalent to the development of w up to the $2n$th order and to satisfy

$$\rho_n^{-4} = \frac{d^2\rho_{n-1}}{dt^2} / \rho_{n-1} + \omega^2. \tag{20}$$

The ρ_n series is infinite and contains some parts of the higher order corrections of w. We consequently get a new way to compute the adiabatic invariants by replacing w by ρ_n, in Eq. (8), so that

$$J_n = \frac{1}{2}\left[\frac{q^2}{\rho_n^2} + \left(\rho_n \cdot \frac{dq}{dt} - q \cdot \frac{d\rho n}{dt}\right)^2\right]. \tag{21}$$

C. *Comparative Study of the Three Methods*

If in the expression of I given by (8) we replace w by its $2n$th order approximation we get the Lewis adiabatic invariant I_n. Now expanding I_n in powers of ϵ, we obtain an infinite series (because of the $1/w3$ term). As has been shown for the zero order, if we keep the $2n + 1$ first terms of this series we recover strictly the Kulsrud adiabatic invariant of order $2n + 1$. Consequently we can write

$$I_n = L_{2n+1} + \mathcal{O}(\epsilon^{2n+2}).$$

$\mathcal{O}(\epsilon^{2n+2})$ stands for a sum of terms going to zero at least as fast as ϵ^{2n+2}.

Now ρ_n is equivalent to the development of w up to the $2n$th order in ϵ the same relation can be writhen for J_n

$$J_n = L_{2n+1} + \mathcal{O}(\epsilon^{2n+2}).$$

Therefore, if ϵ is small enough the three methods are equivalent, but the series I_n, J_n, and L_{2n+1} have different radius of convergence.

Let us remark that the Lewis and the Chandrasekhar methods give expansion in ϵ^2 so that computing one more term in the series is equivalent to computing two orders more in Kulsrud series.

Finally let us point out that our proposed method [getting ρ_n from the recurrence formula (20) and introducing this value in the expression (8) of the exact invariant] is the *least* analytical but the *most* suitable to numerical computation. We do not need the knowledge of analytical expression for L_n or w. The only work to be done is a recurrent differentiation of (20). We show in the next section how we do it.

3. Computational Procedure

Using a development of Eq. (9) Lewis [2] gives the expression of w up to ϵ^6 which is equivalent to ρ_3. From a computational point of view (20) is much more interesting. We carry out successive derivations and do not have to expand w^{-3} but only to manipulate a recurrence formula.

In the Lewis method analytic expressions of the quantities w_{2n} are needed in order to build w_{2n+2}, and similarly in the Kulsrud method we must have an analytic expression for each coefficient. The interest of our iterative method is that we can work out numerical algorithms to get the higher order terms. (It is well known that for numerical computations iterative formulas are very useful.)

Instead of computing ρ_n through Eq. (20) it is more convenient to use the expression,

$$R_n = \rho_n^{-4} = (\omega\omega_1 \cdots \omega_n)^2 \tag{22}$$

getting rid of the fractional powers. Equation (20) becomes

$$R_{j+1} = \frac{5}{16} R_j^{-2} \left(\frac{dR_j}{dt}\right)^2 - \frac{1}{4} R_j^{-1} \frac{d^2R_j}{dt^2} + \omega^2, \tag{23}$$

We write (23) under the form,

$$R_{j+1} = f_0(R_j, \dot{R}_j, \ddot{R}_j, \omega), \tag{24}$$

where the dots indicate the derivative with repect to time.

The successive derivatives of (24) can be formally obtained. However, their determinations become intricate very rapidly. Consequently, an algebraic language such as FORMAC can be used,

$$\dot{R}_{j+1} = \frac{\partial f_0}{\partial R_j} \cdot \dot{R}_j + \frac{\partial f_0}{\partial \dot{R}_j} \ddot{R}_j + \frac{\partial f_0}{\partial \ddot{R}_j} \cdot \dddot{R}_j + \frac{\partial f_0}{\partial \omega} \dot{\omega}. \tag{25}$$

Equation (25) is formally written

$$\dot{R}_{j+1} = f_1(R_j, \dot{R}_j, \ddot{R}_j, \dddot{R}_j, \omega, \dot{\omega}). \tag{26}$$

The derivation is carried on

$$\ddot{R}_{j+1} = f_2(R_j, \dot{R}_j, \ddot{R}_j, \dddot{R}_j, R_j^{(4)}, \omega, \dot{\omega}, \ddot{\omega}).$$

The $(2n-2)$ derivative, i.e., $d^{2n-2}(R_{j+1})/dt^{2n-2}$, is, therefore, a function of

$$R_j, \dot{R}_j, \ldots, R_j^{(2n-1)}, R_j^{(2n)}, \omega, \dot{\omega}, \ddot{\omega}, \ldots, \omega^{(2n-2)}.$$

It must be pointed out that the algebraic language gives the formal expressions:

$$\frac{\partial f_0}{\partial R_j};\frac{\partial f_0}{\partial \dot{R}_j};\frac{\partial f_0}{\partial \ddot{R}_j};\frac{\partial f_1}{\partial R_j};\frac{\partial f_1}{\partial \dot{R}_j};\frac{\partial f_1}{\partial \ddot{R}_j};\ldots$$

The expressions

$$\frac{\partial f_0}{\partial \omega},\frac{\partial f_1}{\partial \omega},\frac{\partial f_1}{\partial \dot{\omega}},\ldots$$

are quite trivial and do not need the help of a computer. Once this algebraic part is done, the calculations will be entirely numerical.

We have now to deal with $2n - 2$ formal expressions. Starting with R_0 (that is to say ω^2) and its $2n$ first derivatives numerical values at time t_0, we can compute on $R_1(t_0)$, $\dot{R}_1(t_0)$... $R_1^{(2n-2)}(t_0)$.

Introducing these new values in the same formal expressions, we get $R_2(t_0)$, $\dot{R}_2(t_0)$,..., $R_2^{(2n-4)}(t_0)$, etc. Thus, at the end of the process, we get numerical values of R_1, R_2, R_3,..., R_n at time t_0.

4. Application of the Adiabatic Invariant in Two Particular Cases

A. *Only Adiabatic Effects are Present*

Choose (see Fig. 1)

$$\omega(t) = (\epsilon t)^{-2n/2n+1}. \tag{27}$$

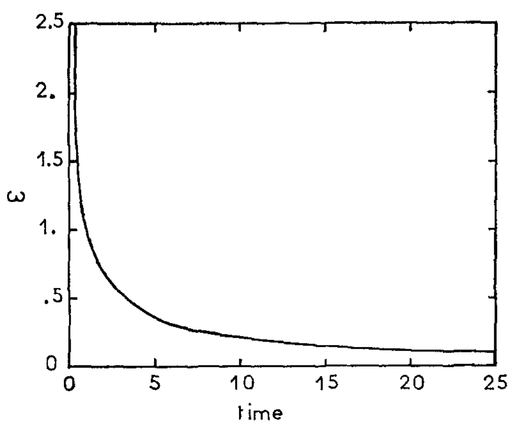

FIG. 1. Function $\omega(t) = t^{-2/3}$.

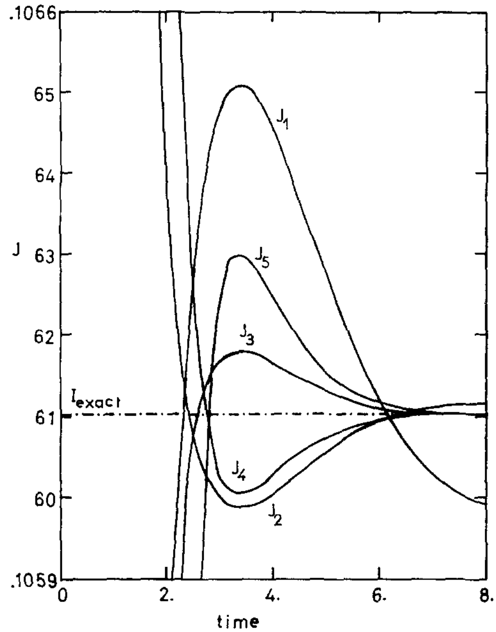

FIG. 2. Adiabatic invariant J_1–J_5 with $\omega(t) = t^{-2/3}$ for time $2 < t < 8$. Note the asymptoticity of the series.

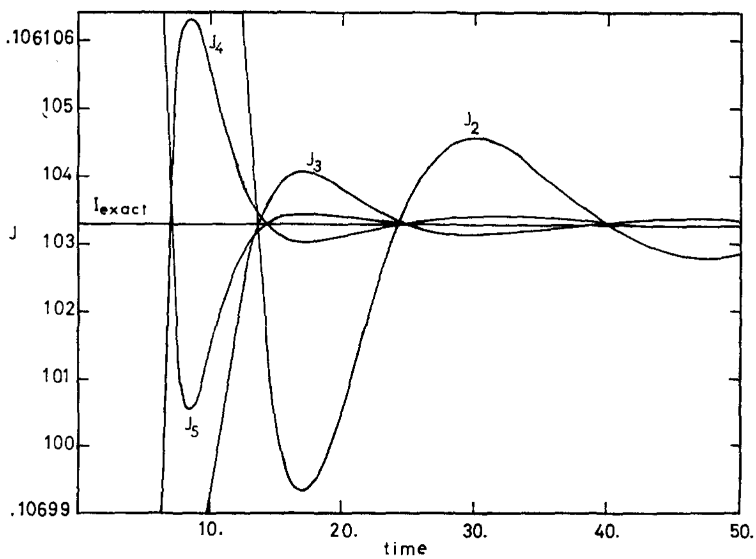

FIG. 3. Adiabatic invariant J_2–J_5 with $\omega(t) = t^{-2/3}$ for time $t < 50$. We are in good adiabatic conditions and the series converges absolutely.

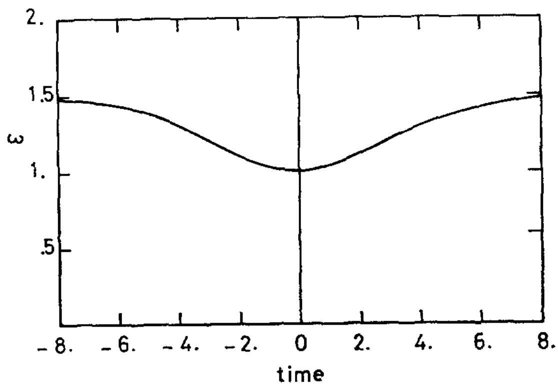

FIG. 4. Function $\omega(t) = 1.5\text{-}0.5 \exp -T^2$.

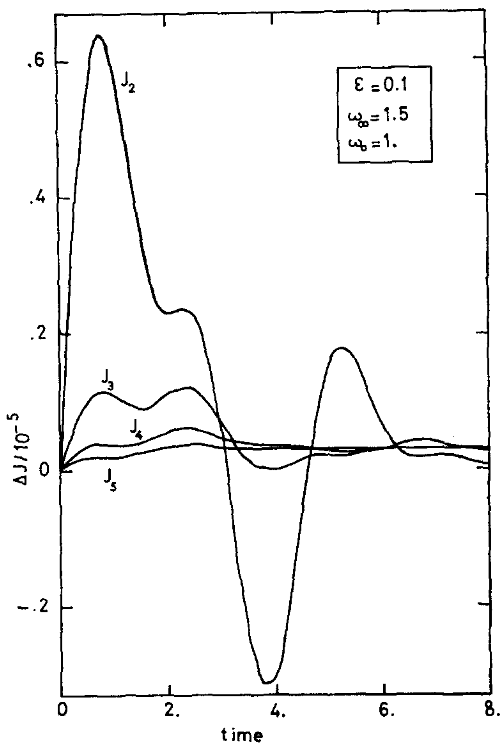

FIG. 5. Adiabatic invariant J_2–J_5 for time $0 < t < 8$.

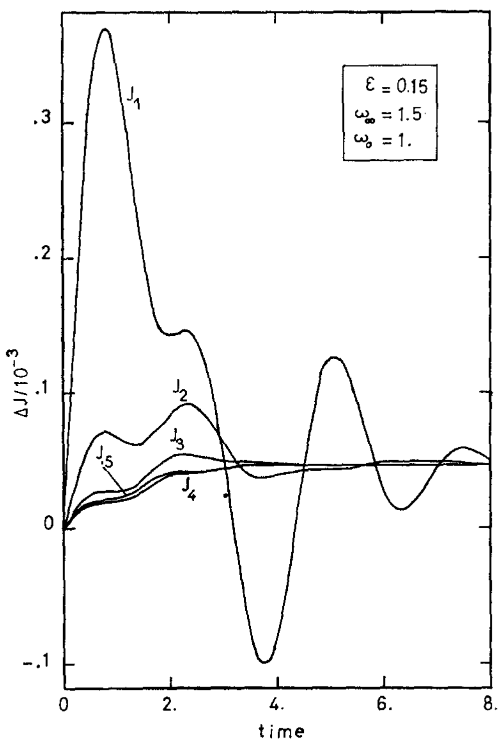

FIG. 6. Adiabatic invariant J_1–J_5 for time $0 < t < 8$.

It has been shown [2] that there is an exact solution of (6) which is

$$w = t^{n/2n+1} \mid G(t, \epsilon)|^2,$$

where

$$G(t, \epsilon) = \left[\sum_0^n (-1)^k \frac{(n+k)!}{k!(n-k)!} \left(\frac{\epsilon}{2i\,(2n+1)}\right)^k t^{-k/(2n+1)}\right]^{1/2}. \tag{28}$$

$| G(t, \epsilon)|^2$ can be expanded in terms of ϵ^{2k} so that we get the Lewis expansion for w. Figs. 2 and 3, show the results for $\epsilon = 1$ with $n = 1$, i.e., for $\omega(t) = t^{-2/3}$. Because of the divergence of ω for $t = 0$ this case has no physical meaning but is interesting because the exact solution of Eq. (6) and the value of the exact invariant can be obtained. See [2].

Another interesting point is the difference of behavior in the terms of the series for different times. In Fig. 2 t goes from 0 to 8 and in that interval the adiabaticity conditions $\dot{\omega}/\omega^2 \ll 1$ are not fulfilled. For time $t < 2.5$ all the invariants blow up on the figure due to the fine scale used. For $2.5 < t < 6$, J_3 is the best approximation. But for a larger time (Fig. 3) the frequency variation is adiabatic, and the convergence of the J_n series is confirmed by the numerical calculations. The exact invariant towards which J_n converges can be very precisely obtained. It must be pointed out that in this case there is no nonadiabatic jump, i.e., that the final value of the adiabatic invariant is strictly equal to the value of the exact invariant.

This can be very simply explained. In that case (28) shows that $w(t) = t^{1/3}(1 - t^{-2/3}/9)^{1/2}$ and will converge for any time $t > 1/27$. It must be noticed that our series seems to require a much bigger value of t for convergence.

B. *Nonadiabatic Case*

Figs. 5–8 show the results for the function,

$$\omega(t) = \omega_\infty - (\omega_\infty - \omega_0)\exp(-\epsilon^2 t^2),$$

with $\omega_\infty = 1.5$, $\omega_0 = 1.0$ and different values of $\epsilon(0.1, 0.15, 0.20, 0.25)$.

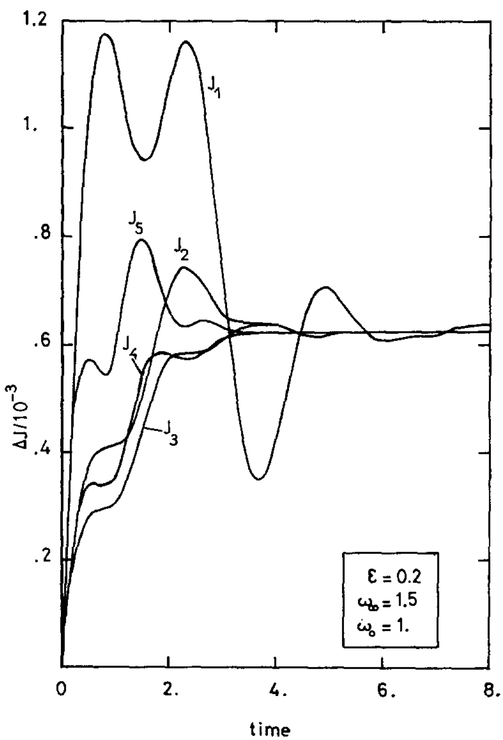

FIG. 7. Adiabatic invariant J_1–J_5 for time $0 < t < 8$.

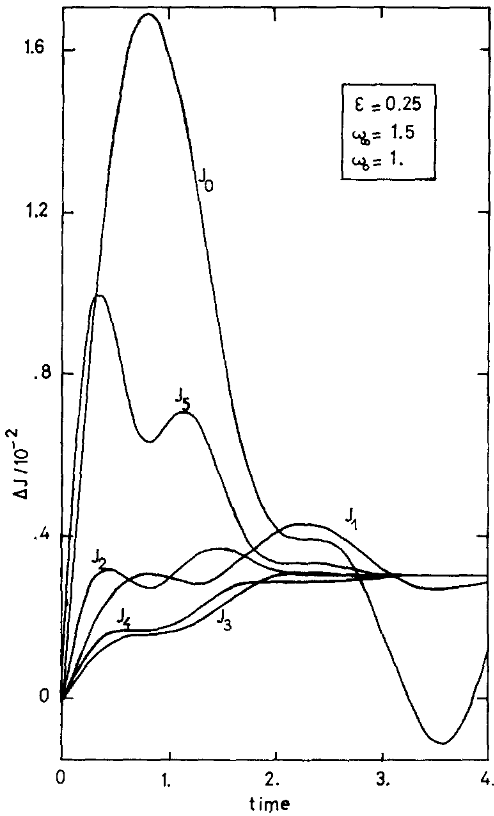

FIG. 8. Adiabatic invariant J_0–J_5 for time $0 < t < 8$.

This example has been studied by Howard [5] who gives numerical calculations for $L_2 = K_0 + \epsilon K_1 + \epsilon^2 K_2$. As already pointed out our J_0 is roughly equivalent to his $K_0 + \epsilon K_1$ approximation while our J_1 goes up to order 3 in ϵ.

Figures 5–8 show the ΔJ curves where $\Delta J = J_n - I$ versus time for different n. We need a very fine scale to exhibit on the figures the variation ΔJ. For $\epsilon = 0.1$ (very good adiabatic conditions) Fig. 5 shows the unavoidable nonadiabatic jump. While J_0 and J_1 are completely out of scale (one should notice the very small (10^{-5}) relative variation for the entire plot) J_5 is the best curve smoothly varying from the initial to the final one without any oscillation. For $\epsilon = 0.15$ (Fig. 6) we must point out

(a) a much larger ordinate scale (10^{-3} in relative value)

(b) J_4 and J_5 are the best curves, J_5 presenting no marked improvement upon J_4. For $\epsilon = 0.2$ (Fig. 7) the asymptotic nature of the series is confirmed.

J_3 and J_4 show the smoothest variation, J_5 is definitively worse, and a "giant" nonadiabatic jump (compared to the preceeding one) is obtained. Finally the case $= 0.25$ (Fig. 8) is similar. Notice that J_0 now appears on the figure. This jump has been shown to decrease with ϵ as $\exp - \eta/\epsilon$, where η is a constant [5, 6]. This means that the jump goes to zero faster than any power of ϵ. This is in agreement with both Howard and our results. We found $\eta = 1.52$.

5. Solution of the Harmonic Oscillator Equation

What is the use of the high-order adiabatic invariants? The answer is suggested in Fig. 5. We see that the constancy of J_n is an indication of how the ρ_n approximates w, one of the exact solutions of Eq. (6). Consequently, if we are interested in solving Eq. (1) it is important to get an approximation of w valid for all times. This can be seen on Eq. (5), where the new variable θ is shown to be the integral of w^{-2}. Consequently, although for time $t = 8$, we see that the differences between J_3, J_4, and J_5 are bigger than the unavoidable nonadiabatic jump, we can predict that the large variation of J_3 between $t = 0$ and $t = 3$ will give a solution much worse than J_4 and J_5. This last quantity just experiences the nonadiabatic jump without any further variation.

To check this idea the solution of (1) was obtained in the following way. ρ_n was

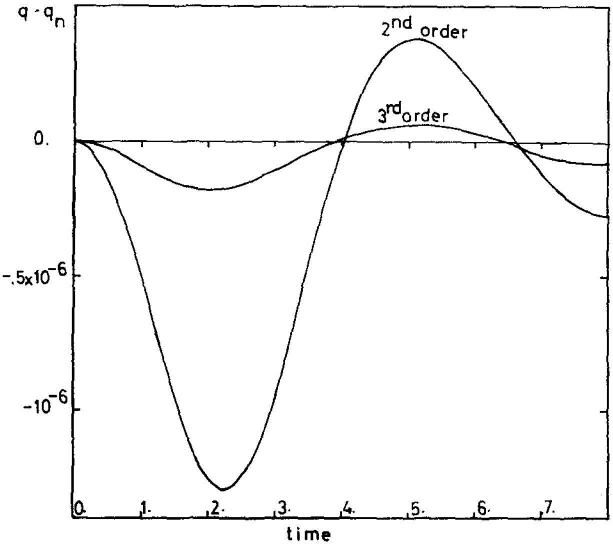

FIG. 9. Exact and approximate solutions, q_2 and q_3, $\omega(t) = 1.5 - .5 \exp - t^2/100$.

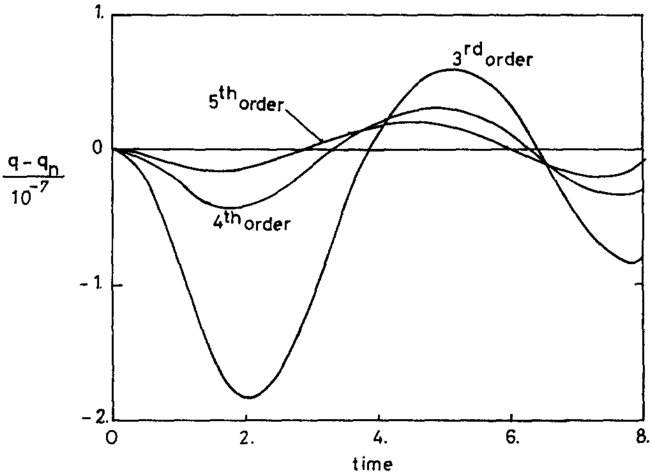

FIG. 10. Exact and approximate solutions, q_3, q_4, and q_5, $\omega(t) = 1.5 - .5 \exp -t^2/100$.

computed as previously indicated and the approximation, $w = \rho_n$, was introduced in (4), (5), and (6). More precisly we have

$$\theta_n(t) = \int_0^t \frac{1}{\rho_n{}^2}\,dt',$$
$$Q_n = A_n \cos\theta_n + B_n \sin\theta_n . \tag{29}$$

Where A_n and B_n are determined from the initial conditions on q_0 and $\dot{q}_0$ through the relations,

$$Q_n(0) = A_n = q_0/\rho_n(0),$$
$$(dQ_n/d\theta_n)(0) = B_n = \rho_n(0)\,\dot{q}_0 - \dot{\rho}_n(0)\,q_0 .$$

For the value of the parameters indicated on Fig. 5; we have computed (29) with a numerical integration scheme involving only 15 points for the entire interval, 0–8. Figures 9 and 10 indicate the difference between the different approximations and the exact solutions obtained by a Runge Kutta method. Indeed the improvement is still important when we go from ρ_4 to ρ_5. Of course, the difference is very small and the computation of high order ρ_n is only justified if we need a great accuracy. This can be the case in some astronomical problem (or motion of particles in nearly periodic orbit—for example, due to a slow variation in the magnetic space field).

6. Conclusion

In this paper we showed how the technical problem of computing high-order adiabatic invariants have been solved (up to order ten in ϵ). However, it must be pointed out that in most cases the asymptotic convergence of that series raises important problems.

When ϵ is very small, the final value is rapidly obtained and the higher-order contributions are quite negligible unless very precise results are desired. On the other hand, if ϵ is not very small the series very quickly becomes divergent (at order 2 or 3), no improvement can be expected, and the obtained high-order terms are not very useful.

These difficulties with adiabatic series are due to the fact that we want to get a valid solution for all times. If we give up this ambitious goal and content ourselves with a solution valid from time 0–T (where T is somewhat related to the small parameter by the relation $T \ll 1/\epsilon$) we can get a formula much more interesting from a numerical point of view.

The idea is to use an expansion in the small parameter ϵ with $X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots$. Now the initial conditions are fully absorbed by the zero order term X_0, i.e., $X(0) = X_0(0)$ and $\dot{X}(0) = \dot{X}_0(0)$ the higher orders are computed taking carefully into account initial conditions $X_1(0) = X_2(0) = \ldots = \dot{X}_1(0) = \dot{X}_2(0) = \ldots = 0$.

Secular terms can arise but they will in the worst case vary $(\epsilon t)^n$ and, consequently, provided we take $\epsilon T \ll 1$, they will not give us any trouble. After this "step" T the process is repeated the value of $X(T)$ and $\dot{X}(T)$ being again absorbed by X_0 and $\dot{X}_0$. This is identical to a classical numerical scheme but with "giant steps" and fully taking into account the possibilities of treating analytically some part of the equations. Such algebraic numerical method has been used for the Mathieu equations and has given very encouraging results.

Appendix — Algebraic Computation

As we have already pointed out we compute only the successive derivatives of the expression,

$$V = \frac{5}{16} R^{-2} \left(\frac{dR}{dt}\right)^2 - \frac{1}{4} R^{-1} \frac{d^2R}{dt^2}.$$

It can be easily seen that these derivatives can be written under the following form:

$$\frac{d^k V}{at} = \sum_{i=1}^{k+2} ZC(k,j)/R^j.$$

The FORMAC program computes the algebraic expression of the $ZC(k, j)$ which obviously are functions of the R derivatives. The obtained formulas for ZC, which permit the obtention of the ten first derivatives of the recurrence formula (and consequently the numerical computation of the R_1, R_2,..., R_6), are shown hereafter.

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ZC(1,1) = -D(3)*(1/4 )
ZC(1,2) = D(2)*D(1)*(7/8 )
ZC(1,3) = -D(1)**3*(5/8 )

ZC(2,1) = -D(4)*(1/4 )
ZC(2,2) = D(3)*D(1)*(9/8 ) + D(2)**2*(7/8 )
ZC(2,3) = -D(2)*D(1)**2*(29/8 )
ZC(2,4) = D(1)**4*(15/8 )


ZC(3,1) = -D(5)*(1/4 )
ZC(3,2) = D(4)*D(1)*(11/8 ) + D(3)*D(2)*(23/8 )
ZC(3,3) = -D(3)*D(1)**2*(47/8 )-D(2)**2*D(1)*9
ZC(3,4) = D(2)*D(1)**3*(147/8 )
ZC(3,5) = -D(1)**5*(15/2 )

ZC(4,1) = -D(6)*(1/4 )
ZC(4,2) = D(5)*D(1)*(13/8 ) + D(4)*D(2)*(17/4 ) + D(3)**2*(23/8 )
ZC(4,3) = -D(4)*D(1)**2*(69/8 )-D(3)*D(2)*D(1)*(71/2 )-D(2)**3*9
ZC(4,4) = D(3)*D(1)**3*36 + D(2)**2*D(1)**2*(657/8 )
ZC(4,5) = -D(2)*D(1)**4*111
ZC(4,6) = D(1)**6*(75/2 )


ZC(5,1) = -D(7)*(1/4 )
ZC(5,2) = D(6)*D(1)*(15/8 ) + D(5)*D(2)*(47/8 ) + D(4)*D(3)*10
ZC(5,3) = -D(5)*D(1)**2*(95/8 )-D(4)*D(2)*D(1)*(245/4 )-D(3)*D(2)
**2*(125/2 )-D(3)**2*D(1)*(165/4 )
ZC(5,4) = D(4)*D(1)**3*(495/8 ) + D(3)*D(2)*D(1)**2*(1515/4 ) + D
(2)**3*D(1)*(765/4 )
ZC(5,5) = -D(3)*D(1)**4*255-D(2)**2*D(1)**3*(1545/2 )
ZC(5,6) = D(2)*D(1)**5*780
ZC(5,7) = -D(1)**7*225

ZC(6,1) = -D(8)*(1/4 )
ZC(6,2) = D(7)*D(1)*(17/8 ) + D(6)*D(2)*(31/4 ) + D(5)*D(3)*(127/
8 ) + D(4)**2*10
ZC(6,3) = -D(6)*D(1)**2*(125/8 )-D(5)*D(2)*D(1)*(387/4 )-D(4)*D(3
)*D(1)*(655/4 )-D(4)*D(2)**2*(495/4 )-D(3)**2*D(2)*(665/4 )
ZC(6,4) = D(5)*D(1)**3*(195/2 ) + D(4)*D(2)*D(1)**2*(5985/8 ) + D
(3)*D(2)**2*D(1)*(6075/4 ) + D(3)**2*D(1)**2*(1005/2 ) + D(2)**4*
(765/4 )
ZC(6,5) = -D(4)*D(1)**4*(1005/2 )-D(3)*D(2)*D(1)**3*4080-D(2)**3*
D(1)**2*(6165/2 )
ZC(6,6) = D(3)*D(1)**5*2055 + D(2)**2*D(1)**4*(15525/2 )
ZC(6,7) = -D(2)*D(1)**6*6255
ZC(6,8) = D(1)**8*1575


ZC(7,1) = -D(9)*(1/4 )
ZC(7,2) = D(8)*D(1)*(19/8 ) + D(7)*D(2)*(79/8 ) + D(6)*D(3)*(189/
8 ) + D(5)*D(4)*(287/8 )
ZC(7,3) = -D(7)*D(1)**2*(159/8 )-D(6)*D(2)*D(1)*(287/2 )-D(5)*D(3
)*D(1)*(1169/4 )-D(5)*D(2)**2*(441/2 )-D(4)*D(3)*D(2)*(2975/4 )-D
(4)**2*D(1)*(735/4 )-D(3)**3*(665/4 )
```

```
ZC(7,4) = D(6)*D(1)**3*(1155/8 ) + D(5)*D(2)*D(1)**2*(10647/8 )
 + D(4)*D(3)*D(1)**2*(17955/8 ) + D(4)*D(2)**2*D(1)*(13545/4 ) +
D(3)*D(2)**3*(9135/4 ) + D(3)**2*D(2)*D(1)*(18165/4 )
ZC(7,5) = -D(5)*D(1)**4*(1785/2 )-D(4)*D(2)*D(1)**3*(18165/2 )-D(
3)*D(2)**2*D(1)**2*(55125/2 )-D(3)**2*D(1)**3*6090-D(2)**4*D(1)*
6930
ZC(7,6) = D(4)*D(1)**5*(9135/2 ) + D(3)*D(2)*D(1)**4*46200 + D(2)
**3*D(1)**3*(92925/2 )
ZC(7,7) = -D(3)*D(1)**6*18585-D(2)**2*D(1)**5*84105
ZC(7,8) = D(2)*D(1)**7*56385
ZC(7,9) = -D(1)**9*12600

ZC(8,1) = -D(10)*(1/4 )
ZC(8,2) = D(9)*D(1)*(21/8 ) + D(8)*D(2)*(49/4 ) + D(7)*D(3)*(67/2
 ) + D(6)*D(4)*(119/2 ) + D(5)**2*(287/8 )
ZC(8,3) = -D(8)*D(1)**2*(197/8 )-D(7)*D(2)*D(1)*203-D(6)*D(3)*D(1
)*483-D(6)*D(2)**2*364-D(5)*D(4)*D(1)*(1463/2 )-D(5)*D(3)*D(2)*
1477-D(4)*D(3)**2*(2485/2 )-D(4)**2*D(2)*(1855/2 )
ZC(8,4) = D(7)*D(1)**3*204 + D(6)*D(2)*D(1)**2*(4389/2 ) + D(5)*D
(3)*D(1)**2*4452 + D(5)*D(2)**2*D(1)*(13419/2 ) + D(4)*D(3)*D(2)*
D(1)*22575 + D(4)*D(2)**3*5670 + D(4)**2*D(1)**2*(22365/8 ) + D(3
)**2*D(2)**2*(22785/2 ) + D(3)**3*D(1)*5040
ZC(8,5) = -D(6)*D(1)**4*1470-D(5)*D(2)*D(1)**3*17976-D(4)*D(3)*D(
1)**3*30240-D(4)*D(2)**2*D(1)**2*68355-D(3)*D(2)**3*D(1)*91980-D(
3)**2*D(2)*D(1)**2*91560-D(2)**5*6930
ZC(8,6) = D(5)*D(1)**5*9030 + D(4)*D(2)*D(1)**4*114450 + D(3)*D(2
)**2*D(1)**3*462000 + D(3)**2*D(1)**4*76650 + D(2)**4*D(1)**2*(
348075/2 )
ZC(8,7) = -D(4)*D(1)**6*45990-D(3)*D(2)*D(1)**5*556920-D(2)**3*D(
1)**4*699300
ZC(8,8) = D(3)*D(1)**7*186480 + D(2)**2*D(1)**6*983430
ZC(8,9) = -D(2)*D(1)**8*564480
ZC(8,10) = D(1)**10*113400


ZC(9,1) = -D(11)*(1/4 )
ZC(9,2) = D(10)*D(1)*(23/8 ) + D(9)*D(2)*(119/8 ) + D(8)*D(3)*(
183/4 ) + D(7)*D(4)*93 + D(6)*D(5)*(525/4 )
ZC(9,3) = -D(9)*D(1)**2*(239/8 )-D(8)*D(2)*D(1)*(1107/4 )-D(7)*D(
3)*D(1)*753-D(7)*D(2)**2*567-D(6)*D(4)*D(1)*(2667/2 )-D(6)*D(3)*D
(2)*2688-D(5)*D(4)*D(2)*(8127/2 )-D(5)*D(3)**2*(5439/2 )-D(5)**2*
D(1)*(3213/4 )-D(4)**2*D(3)*(6825/2 )
ZC(9,4) = D(8)*D(1)**3*(2223/8 ) + D(7)*D(2)*D(1)**2*(6831/2 ) +
D(6)*D(3)*D(1)**2*(16191/2 ) + D(6)*D(2)**2*D(1)*(24381/2 ) + D(5
)*D(4)*D(1)**2*(48951/4 ) + D(5)*D(3)*D(2)*D(1)*49329 + D(5)*D(2)
**3*(24759/2 ) + D(4)*D(3)*D(2)**2*62370 + D(4)*D(3)**2*D(1)*(
82845/2 ) + D(4)**2*D(2)*D(1)*(123795/4 ) + D(3)**3*D(2)*27825
ZC(9,5) = -D(7)*D(1)**4*2286-D(6)*D(2)*D(1)**3*32634-D(5)*D(3)*D(
1)**3*65024-D(5)*D(2)**2*D(1)**2*149121-D(4)*D(3)*D(2)*D(1)**2*
500850-D(4)*D(2)**3*D(1)*251370-D(3)*D(2)**4*126630-D(4)**2*D(1)
**3*(82845/2 )-D(3)**2*D(2)**2*D(1)*504630-D(3)**3*D(1)**2*111720
ZC(9,6) = D(6)*D(1)**5*16380 + D(5)*D(2)*D(1)**4*249480 + D(4)*D(
3)*D(1)**4*418950 + D(4)*D(2)**2*D(1)**3*1261575 + D(3)*D(2)**3*D
(1)**2*2542050 + D(3)**2*D(2)*D(1)**3*1688400 + D(2)**5*D(1)*
382725
ZC(9,7) = -D(5)*D(1)**6*100170-D(4)*D(2)*D(1)**5*1519560-D(3)*D(2
)**2*D(1)**4*7654500-D(3)**2*D(1)**5*1016820-D(2)**4*D(1)**3*
```

```
3841425
ZC(9,8) = D(4)*D(1)**7*508410 + D(3)*D(2)*D(1)**6*7170660 + D(2)
**3*D(1)**5*10795680
ZC(9,9) = -D(3)*D(1)**8*2056320-D(2)**2*D(1)**7*12383280
ZC(9,10) = D(2)*D(1)**9*6214320
ZC(9,11) = -D(1)**11*1134000

ZC(10,1) = -D(12)*(1/4 )
ZC(10,2) = D(11)*D(1)*(25/8 ) + D(10)*D(2)*(71/4 ) + D(9)*D(3)*(
485/8 ) + D(8)*D(4)*(555/4 ) + D(7)*D(5)*(897/4 ) + D(6)**2*(525/
4 )
ZC(10,3) = -D(10)*D(1)**2*(285/8 )-D(9)*D(2)*D(1)*(1465/4 )-D(8)*
D(3)*D(1)*(4485/4 )-D(8)*D(2)**2*(3375/4 )-D(7)*D(4)*D(1)*(4545/2
 )-D(7)*D(3)*D(2)*4575-D(6)*D(5)*D(1)*(6405/2 )-D(6)*D(4)*D(2)*
8085-D(6)*D(3)**2*(10815/2 )-D(5)*D(4)*D(3)*(32655/2 )-D(5)**2*D(
2)*(19467/4 )-D(4)**3*(6825/2 )
ZC(10,4) = D(9)*D(1)**3*(735/2 ) + D(8)*D(2)*D(1)**2*(40635/8 )
 + D(7)*D(3)*D(1)**2*13770 + D(7)*D(2)**2*D(1)*(41445/2 ) + D(6)*
D(4)*D(1)**2*(97335/4 ) + D(6)*D(3)*D(2)*D(1)*97965 + D(6)*D(2)**
3*24570 + D(5)*D(4)*D(2)*D(1)*(295785/2 ) + D(5)*D(3)*D(2)**2*(
297675/2 ) + D(5)*D(3)**2*D(1)*98910 + D(4)*D(3)**2*D(2)*(499275/
2 ) + D(5)**2*D(1)**2*(29295/2 ) + D(4)**2*D(3)*D(1)*(496125/4 )
 + D(4)**2*D(2)**2*(373275/4 ) + D(3)**4*27825
ZC(10,5) = -D(8)*D(1)**4*(6795/2 )-D(7)*D(2)*D(1)**3*55440-D(6)*D
(3)*D(1)**3*131040-D(6)*D(2)**2*D(1)**2*295785-D(5)*D(4)*D(1)**3*
197820-D(5)*D(3)*D(2)*D(1)**2*1194480-D(5)*D(2)**3*D(1)*599130-D(
4)*D(3)*D(2)**2*D(1)*3014550-D(4)*D(3)**2*D(1)**2*1001700-D(4)*D(
2)**4*378000-D(4)**2*D(2)*D(1)**2*(1497825/2 )-D(3)**2*D(2)**3*
1011150-D(3)**3*D(2)*D(1)*1344000
ZC(10,6) = D(7)*D(1)**5*27810 + D(6)*D(2)*D(1)**4*494550 + D(5)*D
(3)*D(1)**4*998550 + D(5)*D(2)**2*D(1)**3*3005100 + D(4)*D(3)*D(2
)*D(1)**3*10080000 + D(4)*D(2)**3*D(1)**2*7583625 + D(3)*D(2)**4*
D(1)*7630875 + D(4)**2*D(1)**4*(1252125/2 ) + D(3)**2*D(2)**2*D(1
)**2*15214500 + D(3)**3*D(1)**3*2247000 + D(2)**6*382725
ZC(10,7) = -D(6)*D(1)**6*198450-D(5)*D(2)*D(1)**5*3617460-D(4)*D(
3)*D(1)**5*6066900-D(4)*D(2)**2*D(1)**4*22821750-D(3)*D(2)**3*D(1
)**3*61236000-D(3)**2*D(2)*D(1)**4*30523500-D(2)**5*D(1)**2*
13820625
ZC(10,8) = D(5)*D(1)**7*1209600 + D(4)*D(2)*D(1)**6*21366450 + D(
3)*D(2)**2*D(1)**5*128992500 + D(3)**2*D(1)**6*14288400 + D(2)**4
*D(1)**4*30868375
ZC(10,9) = -D(4)*D(1)**8*6123600-D(3)*D(2)*D(1)**7*98582400-D(2)
**3*D(1)**6*173048400
ZC(10,10) = D(3)*D(1)**9*24721200 + D(2)**2*D(1)**8*167378400
ZC(10,11) = -D(2)*D(1)**10*74617200
ZC(10,12) = D(1)**12*12474000
```

NOTATIONS OF FORTRAN

D(j) means $\frac{d^jR}{dt^j}$

* indicates a multiplication

* * indicates an exponentiation

For example Z C (2,2) should be read :

$$Z\,C\,(2,2) = \frac{9}{8}\;\frac{d^3R}{dt^3}\;\frac{dR}{dt} + \frac{7}{8}\left(\frac{d^2R}{dt^2}\right)^2$$